

FLAT PSEUDO-RIEMANNIAN HOMOGENEOUS SPACES WITH NON-ABELIAN HOLONOMY GROUP

OLIVER BAUES AND WOLFGANG GLOBKE

ABSTRACT. We construct homogeneous flat pseudo-Riemannian manifolds with non-abelian fundamental group. In the compact case, all homogeneous flat pseudo-Riemannian manifolds are complete and have abelian linear holonomy group. To the contrary, we show that there do exist non-compact and non-complete examples, where the linear holonomy is non-abelian, starting in dimensions ≥ 8 , which is the lowest possible dimension. We also construct a complete flat pseudo-Riemannian homogeneous manifold of dimension 14 with non-abelian linear holonomy. Furthermore, we derive a criterion for the properness of the action of an affine transformation group with transitive centralizer.

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1. INTRODUCTION

A flat pseudo-Riemannian manifold M is called homogeneous if its isometry group acts transitively. As examples show [2, 4], non-compact flat pseudo-Riemannian homogeneous manifolds are not necessarily complete. The study of complete flat homogeneous pseudo-Riemannian manifolds was pioneered by Wolf in a series of papers [8, 9, 10]. Such manifolds are isometric to a manifold of the form $\mathbb{R}^{r,s}/\Gamma$, for some subgroup $\Gamma \leq \text{Iso}(\mathbb{R}^{r,s})$. Homogeneity implies that the centralizer of Γ in $\text{Iso}(\mathbb{R}^{r,s})$ acts transitively on $\mathbb{R}^{r,s}$. One basic observation is that in this situation the group Γ is nilpotent of nilpotency class at most two. This fact also holds for the holonomy group Γ of a non-complete homogeneous pseudo-Riemannian manifold. Apparently, it was believed for some time that Γ , or, which is considerably weaker, the linear part of Γ should be abelian. However, as observed in [1] non-abelian fundamental groups Γ appear for compact complete flat homogeneous pseudo-Riemannian manifolds.

In this note, we present some additional new results on the structure of flat pseudo-Riemannian homogeneous manifolds. Although non-abelian fundamental groups Γ do appear, in the compact case the linear holonomy is always abelian. In addition, we show that every homogeneous flat pseudo-Riemannian manifold of dimension less than eight has abelian linear holonomy.

As one of our main results, we give examples of homogeneous manifolds with non-abelian linear holonomy group. We construct an eight-dimensional non-complete manifold U/Γ_1 , where U is an open domain in $\mathbb{R}^{4,4}$, and a fourteen-dimensional complete manifold $\mathbb{R}^{7,7}/\Gamma_2$, both with non-abelian linear holonomy. The groups $\Gamma_1 \leq \text{Iso}(\mathbb{R}^{4,4})$ and $\Gamma_2 \leq \text{Iso}(\mathbb{R}^{7,7})$ are isomorphic to the integral Heisenberg group on two generators and map injectively to their linear parts. These manifolds give the first examples of flat pseudo-Riemannian homogeneous manifolds with non-abelian linear holonomy group.

2. PRELIMINARIES

Here, $\mathbb{R}^{r,s}$ denotes \mathbb{R}^{r+s} endowed with a scalar product $\langle \cdot, \cdot \rangle$ of signature (r, s) , and $\text{Iso}(\mathbb{R}^{r,s})$ its group of isometries. Affine maps of $\mathbb{R}^{r,s}$ are written as $\gamma = (I + A, v)$, where $I + A$ is the linear part (I the identity matrix), and v the translation part.

The groups $\Gamma \subset \text{Iso}(\mathbb{R}^{r,s})$ with transitive centralizer in $\text{Iso}(\mathbb{R}^{r,s})$ were studied first in [8]. We sum up some of the results for later reference. Note that all of the following holds also if the centralizer of Γ is only required to have an open orbit in $\mathbb{R}^{r,s}$ (compare [1, Proposition 3.10] or [3, Lemma 4.1]).

Lemma 2.1. Γ consists of affine transformations $\gamma = (I + A, v)$, where $A^2 = 0$, $v \perp \text{im } A$ and $\text{im } A$ is totally isotropic.

Lemma 2.2. For $\gamma_i = (I + A_i, v_i) \in \Gamma$, $i = 1, 2, 3$, we have $A_1 A_2 v_1 = 0 = A_2 A_1 v_2$, $A_1 A_2 A_3 = 0$ and $[\gamma_1, \gamma_2] = (I + 2A_1 A_2, 2A_1 v_2)$.

Lemma 2.3. If $\gamma = (I + A, v) \in \Gamma$, then $\langle Ax, y \rangle = -\langle x, Ay \rangle$, $\text{im } A = (\ker A)^\perp$, $\ker A = (\text{im } A)^\perp$ and $Av = 0$.

Theorem 2.4. Γ is 2-step nilpotent (meaning $[\Gamma, [\Gamma, \Gamma]] = \{\text{id}\}$).

For $\gamma = (I + A, v) \in \Gamma$, set $\text{hol}(\gamma) = I + A$ (the linear component of γ). We write $A = \log(\text{hol}(\gamma))$.

Definition 2.5. The linear holonomy group of Γ is $\text{hol}(\Gamma) = \{\text{hol}(\gamma) \mid \gamma \in \Gamma\}$.

In the latest edition of the book [7], a characterization of those Γ with abelian linear holonomy is given:

Proposition 2.6. The following are equivalent:

- (1) $\text{hol}(\Gamma)$ is abelian.
- (2) If $(I + A_1, v_1), (I + A_2, v_2) \in \Gamma$, then $A_1 A_2 = 0$.
- (3) The space $U_\Gamma = \sum_{\gamma \in \Gamma} \text{im } A$ is totally isotropic.

The proof of Proposition 2.6 uses only the lemmata above. Thus, the following structure theorem for groups Γ , such that the centralizer of Γ has an open orbit in $\mathbb{R}^{r,s}$, and Γ with abelian linear holonomy, holds:

Theorem 2.7. *If $\text{hol}(\Gamma)$ is abelian, then for every Witt basis with respect to U_Γ (see section 4) and $(I + A, v) \in \Gamma$, the matrix A is of the form*

$$(2.1) \quad A = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where C is a skew-symmetric $k \times k$ -matrix ($k = \dim U_\Gamma$).

In section 3, we show that for compact M the holonomy group $\text{hol}(\Gamma)$ is always abelian, and we present a more refined classification and structure theorem for the groups Γ in the compact case. We give a modification for the structure theorem, Theorem 2.7, which holds for arbitrary Γ , in section 4. In section 5, we show that groups Γ such that $\text{hol}(\Gamma)$ is not abelian exist only for dimensions ≥ 8 , and in section 6 we give an example of such a group. This group does not act freely and therefore cannot be the fundamental group of a complete flat homogeneous pseudo-Riemannian manifold M . But it gives rise to a non-complete example M . We also present an example of a group Γ which acts freely on $\mathbb{R}^{7,7}$ and has a transitive centralizer. This group gives rise to a complete 14-dimensional homogeneous flat pseudo-Riemannian manifold with non-abelian linear holonomy group. To show that the groups Γ involved act properly we derive in section 7 a criterion which shows that a discrete unipotent group acting freely on \mathbb{R}^n , and whose centralizer has an open orbit, acts properly on \mathbb{R}^n .

3. COMPACT FLAT PSEUDO-RIEMANNIAN SPACES

In this section, let M be a compact flat homogeneous pseudo-Riemannian manifold. By [5] (see [1, Corollary 4.5] for an alternative proof), M must be complete. Therefore, $M = \mathbb{R}^{r,s}/\Gamma$, for some group $\Gamma \leq \text{Iso}(\mathbb{R}^{r,s})$ which acts properly discontinuously and freely on $\mathbb{R}^{r,s}$. Let G be the centralizer of Γ in $\text{Iso}(\mathbb{R}^{r,s})$. Since M is homogeneous and compact, then, as follows from [1, Theorem 4.6], G is a nilpotent Lie group which acts simply transitively on $\mathbb{R}^{r,s}$ by isometries. Let $x_0 \in \mathbb{R}^{r,s}$ be a fixed basepoint. There is a unique left invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle_G$ on G such that the orbit map $o : G \rightarrow \mathbb{R}^{r,s}$, $g \mapsto g \cdot x_0$, is an isometry. Moreover, the metric $\langle \cdot, \cdot \rangle_G$ is biinvariant, see [1, Theorem 4.6]. The map o induces an isometry $G/\tilde{\Gamma} \rightarrow M$, where $\tilde{\Gamma}$ is a lattice subgroup of G , which is isomorphic to Γ , and $G/\tilde{\Gamma}$ inherits the pseudo-Riemannian structure from $(G, \langle \cdot, \cdot \rangle_G)$. It also follows that G is (at most) two-step nilpotent (see [1, Lemma 4.8]). Such manifolds $G/\tilde{\Gamma}$ necessarily have abelian linear holonomy group:

Theorem 3.1. *Let G be a Lie group with a biinvariant flat pseudo-Riemannian metric $\langle \cdot, \cdot \rangle_G$, and $\tilde{\Gamma} \leq G$ be a lattice. Then the compact flat pseudo-Riemannian homogeneous manifold $G/\tilde{\Gamma}$ has abelian linear holonomy.*

Proof. Let $\rho : G \rightarrow \text{Iso}(\mathbb{R}^{r,s})$ be the development representation of the right-multiplication of G and put $\Gamma = \rho(\tilde{\Gamma})$. Then, as above, there is an orbit map $o : G \rightarrow \mathbb{R}^{r,s}$, which is an isometry and satisfies $o(g\gamma) = \rho(\gamma)o(g)$. (cf. [1, Proposition 5.2].) This map induces an isometry $G/\tilde{\Gamma} \rightarrow \mathbb{R}^{r,s}/\Gamma$.

Let \mathfrak{g} denote the Lie algebra of G . By [1, Proposition 3.3, Lemma 5.10], the differential of ρ at the identity is equivalent to the affine representation $X \mapsto (\frac{1}{2}\text{ad}(X), X)$ of \mathfrak{g} on the vector space of \mathfrak{g} . In particular, the linear part of the

differential of ρ is equivalent to the adjoint representation ad of \mathfrak{g} . Since \mathfrak{g} is two-step nilpotent, the adjoint representation ad has abelian image. It follows that the linear part of $\rho(G)$ is abelian. Since $\Gamma \leq \rho(G)$, this implies that Γ has abelian linear part. \square

Remark 3.2. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ denote the inner product induced on \mathfrak{g} by $\langle \cdot, \cdot \rangle_G$. Biinvariance of $\langle \cdot, \cdot \rangle_G$ is equivalent to

$$(3.1) \quad \langle [X, Y], Z \rangle_{\mathfrak{g}} = -\langle Y, [X, Z] \rangle_{\mathfrak{g}}$$

Identify $\mathfrak{g} \cong \mathbb{R}^{r,s}$ via the differential of o . Then $\gamma = (I + A, v) = (I + \frac{1}{2}\text{ad}(X), X)$, where $X = \log(\gamma) \in \mathfrak{g}$. Therefore,

$$(3.2) \quad U_{\Gamma} = \sum_{\gamma \in \Gamma} \text{im } A = \sum_{X \in \log(\Gamma)} \text{im } \text{ad}(X) = [\mathfrak{g}, \mathfrak{g}],$$

equals the commutator subalgebra of \mathfrak{g} . (The last equality holds because the elements in $\log(\Gamma)$ generate \mathfrak{g} , since Γ is a lattice in G .) Using biinvariance and 2-step nilpotency, it is easy to see that the space $U_{\Gamma} = [\mathfrak{g}, \mathfrak{g}]$ is totally isotropic. By Theorem 2.7, this is equivalent to $\text{hol}(\Gamma)$ being abelian.

Corollary 3.3. *Let $M = \mathbb{R}^{r,s}/\Gamma$ be a compact flat pseudo-Riemannian homogeneous manifold. Then $\text{hol}(\Gamma)$ is abelian.*

To specify a biinvariant pseudo-Riemannian metric on the Lie group G it is equivalent to construct a biinvariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie algebra of \mathfrak{g} . The metric is flat if and only if \mathfrak{g} is two-step nilpotent. Below, we state a structure theorem for such pairs $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$, taken from [1, Theorem 5.15]. By the above, this yields a structure theorem for groups Γ which are the fundamental groups of compact homogeneous flat pseudo-Riemannian manifolds.

Recall that for an abelian Lie algebra \mathfrak{a} and its dual \mathfrak{a}^* , a Lie product on the space $\mathfrak{a} \oplus \mathfrak{a}^*$ is given by

$$[(X, X^*), (Y, Y^*)] = ([X, Y], \text{ad}^*(X)Y^* - \text{ad}^*(Y)X^* + \omega(X, Y)),$$

where ad^* denotes the coadjoint representation, and $\omega \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ is a 2-cocycle for the adjoint representation. We use the notation $\mathfrak{g} = \mathfrak{a} \oplus_{\omega} \mathfrak{a}^*$ for this Lie algebra. An inner product of split signature on \mathfrak{g} is defined by

$$\langle (X, X^*), (Y, Y^*) \rangle_{\mathfrak{g}} = X^*(Y) + Y^*(X),$$

and it can be shown to be biinvariant if and only if the 3-form $F_{\omega}(X_1, X_2, X_3) = \langle \omega(X_1, X_2), X_3 \rangle$ on \mathfrak{a} is alternating and satisfies

$$F_{\omega}(X_1, [X_2, X_3], X_4) = F_{\omega}(X_2, X_3, [X_1, X_4]),$$

for all $X_i \in \mathfrak{a}$. Then $\mathfrak{a} \oplus_{\omega} \mathfrak{a}^*$ is a 2-step nilpotent Lie algebra with biinvariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Theorem 3.4. *Let \mathfrak{g} be a 2-step nilpotent Lie algebra with biinvariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Then there exists an abelian Lie algebra \mathfrak{a} , an alternating 3-form F_{ω} on \mathfrak{a} and an abelian Lie algebra \mathfrak{z}_0 such that \mathfrak{g} can be written as a direct product of metric Lie algebras*

$$(3.3) \quad \mathfrak{g} = (\mathfrak{a} \oplus_{\omega} \mathfrak{a}^*) \oplus \mathfrak{z}_0.$$

Proof. $[\mathfrak{g}, \mathfrak{g}]$ is an isotropic subspace of \mathfrak{g} . Biinvariance shows that its orthogonal complement $[\mathfrak{g}, \mathfrak{g}]^\perp$ is the center $\mathfrak{z}(\mathfrak{g})$. Let \mathfrak{a} denote the isotropic subspace dual to $[\mathfrak{g}, \mathfrak{g}]$ in \mathfrak{g} (then $[\mathfrak{g}, \mathfrak{g}]$ can be identified with the dual space \mathfrak{a}^* of \mathfrak{a}). Finally, let \mathfrak{z}_0 be a complement of \mathfrak{a}^* in $\mathfrak{z}(\mathfrak{g})$, that is $\mathfrak{z}(\mathfrak{g}) = \mathfrak{a}^* \oplus \mathfrak{z}_0$. Then \mathfrak{z}_0 commutes with and is orthogonal to \mathfrak{a} and \mathfrak{a}^* . So $\mathfrak{g} = (\mathfrak{a} \oplus_\omega \mathfrak{a}^*) \oplus \mathfrak{z}_0$ for some 2-cocycle $\omega \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$. \square

4. STRUCTURE THEOREM

Let $\Gamma \subset \text{Iso}(\mathbb{R}^{r,s})$ such that its centralizer in $\text{Iso}(\mathbb{R}^{r,s})$ has an open orbit in $\mathbb{R}^{r,s}$. For short, we write Δ for the center of Γ . This group is abelian, so it satisfies the conditions of Theorem 2.7. We set $U_\Gamma = \sum_{\gamma \in \Gamma} \text{im } A$, $U_\Delta = \sum_{\gamma \in \Delta} \text{im } A$ and $U_0 = U_\Gamma \cap U_\Gamma^\perp$, which is a totally isotropic subspace. It follows from Lemma 2.3 that

$$(4.1) \quad U_0 = U_\Gamma \cap U_\Gamma^\perp = \sum_{\gamma \in \Gamma} \text{im } A \cap \bigcap_{\gamma \in \Gamma} \ker A.$$

Lemma 4.1. $U_\Delta \perp U_\Gamma$.

Proof. Let $\gamma_1 = (I + A_1, v_1) \in \Delta$ and $\gamma_2 = (I + A_2, v_2) \in \Gamma$. As γ_1 is central, it follows from Lemma 2.2 that $A_1 A_2 = 0$, so $\text{im } A_2 \subset \ker A_1 = (\text{im } A_1)^\perp$ (see Lemma 2.3). Hence $U_\Gamma \perp U_\Delta$. \square

It is easy to see that $U_\Delta \subseteq U_\Gamma$, $U_\Delta \subseteq U_0$, $U_\Delta^\perp \supseteq U_0^\perp \supseteq U_\Gamma^\perp \supseteq U_0 \supseteq U_\Delta$.

Lemma 4.2. $A \cdot U_\Delta^\perp \subseteq U_0$ for all $(I + A, v) \in \Gamma$.

Proof. Let $y \in U_\Delta^\perp$. For all $x \in \mathbb{R}^{r,s}$ and $A, B \in \log(\text{hol}(\Gamma))$,

$$\langle Bx, Ay \rangle = -\langle ABx, y \rangle = 0,$$

by Lemma 2.3 and because AB is central. Hence $Ay \perp U_\Gamma$, that is, $Ay \in U_0$. \square

The following proposition sums up the above:

Proposition 4.3. *The chain of subspaces*

$$\mathbb{R}^{r,s} \supset U_\Delta^\perp \supset U_0 \supset \{0\}$$

is stabilized by $\log(\text{hol}(\Gamma))$ such that each subspace is mapped to the next in the chain.

Given the totally isotropic subspace U_0 , we can find a *Witt basis* for $\mathbb{R}^{r,s}$ with respect to U_0 as follows: If $k = \dim U_0$, there exists a basis for $\mathbb{R}^{r,s}$,

$$(4.2) \quad \{u_1, \dots, u_k, \quad w_1, \dots, w_{n-2k}, \quad u_1^*, \dots, u_k^*\},$$

such that $\{u_1, \dots, u_k\}$ is a basis of U_0 , $\{w_1, \dots, w_{n-2k}\}$ is a basis of a non-degenerate subspace W such that $U_0^\perp = U_0 \oplus W$, and $\{u_1^*, \dots, u_k^*\}$ is a basis of a space U_0^* such that $\langle u_i, u_j^* \rangle = \delta_{ij}$ (then U_0^* is called a *dual space* for U_0). Let \tilde{I} denote the signature matrix representing the restriction of $\langle \cdot, \cdot \rangle$ to W with respect to the chosen basis of W .

The following generalizes Theorem 2.7:

Theorem 4.4. *Let $\gamma = (I + A, v) \in \Gamma$ and fix a Witt basis with respect to U_0 . Then the matrix representation of A in this basis is*

$$(4.3) \quad A = \begin{pmatrix} 0 & -B^\top \tilde{I} & C \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix},$$

with $B \in \mathbb{R}^{(n-2k) \times k}$ and $C \in \mathfrak{so}_k$ (where $k = \dim U_0$). The columns of B are isotropic and mutually orthogonal with respect to \tilde{I} .

Proof. With respect to the given Witt basis, A is represented by a matrix

$$\begin{pmatrix} A_1 & -A_2^\top \tilde{I} & A_3 \\ A_4 & A_5 & A_2 \\ A_6 & -A_4^\top \tilde{I} & -A_1^\top \end{pmatrix}$$

with A_3, A_6 skew-symmetric, $A_5 \in \mathfrak{so}(\tilde{I})$. By Proposition 4.3, $A_1 = 0$, $A_4 = 0$, $A_6 = 0$, and also $A_5 = 0$. Set $B = A_2$, $C = A_3$.

The condition $A^2 = 0$ implies $-B^\top \tilde{I} B = 0$, so all columns of B are isotropic and mutually orthogonal with respect to \tilde{I} . \square

5. DIMENSION BOUNDS FOR NON-ABELIAN HOLONOMY GROUPS

We sum up two rules which have to be satisfied by the representation matrices (4.3). Given matrices A_i ($i = 1, 2$), B_i and C_i refer to the respective matrix blocks in (4.3).

- (1) *Crossover rule:* Given A_1 and A_2 , let b_2^i be a column of B_2 and b_1^k a column of B_1 . Then $\langle b_1^k, b_2^i \rangle = -\langle b_1^i, b_2^k \rangle$. In particular, $\langle b_1^k, b_2^k \rangle = 0$, and $\langle b_1^i, b_1^k \rangle = 0$. If $\langle b_1^i, b_2^k \rangle \neq 0$ then $b_1^k, b_1^i, b_2^k, b_2^i$ are linearly independent. (The product of $A_1 A_2$ contains $-B_1^\top \tilde{I} B_2$ as the skew-symmetric upper right block, so its entries are the values $-\langle b_1^k, b_2^i \rangle$.)
- (2) *Duality rule:* Assume A_1 is not central (that is $A_1 A_2 \neq 0$ for some A_2). Then B_2 contains a column b_2^i and B_1 a column b_1^j such that $\langle b_1^j, b_2^i \rangle \neq 0$.

Theorem 5.1. *Let $\Gamma \subset \text{Iso}(\mathbb{R}^{r,s})$ be a group acting on \mathbb{R}^n , $n = r + s$, whose centralizer in $\text{Iso}(\mathbb{R}^{r,s})$ has an open orbit. If $\text{hol}(\Gamma)$ is non-abelian, then*

$$n \geq 8.$$

As Example 6.2 shows, this is a sharp lower bound.

Proof. If $\text{hol}(\Gamma)$ is not abelian, there exist $\gamma_1 = (I + A_1, v_1), \gamma_2 = (I + A_2, v_2)$ such that $A_1 A_2 \neq 0$ (Lemma 2.2).

Let W be a vector space complement of U_0 in U_0^\perp , so W is non-degenerate and $x \in \mathbb{R}^{r,s}$ can be written $x = u + w + u^*$ with $u \in U_0, w \in W, u^* \in U_0^*$. Then

$$(*) \quad A_1 x = \begin{pmatrix} 0 & -B_1^\top \tilde{I} & C_1 \\ 0 & 0 & B_1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ w \\ u^* \end{pmatrix} = \begin{pmatrix} -B_1^\top \tilde{I} w + C_1 u^* \\ B_1 u^* \\ 0 \end{pmatrix}.$$

By the duality rule, there are columns in B_1, B_2 which are non-orthogonal to one another. Then, by the crossover rule, B_1 and B_2 together contain at least four linearly independent columns. This implies $\dim W \geq 4$.

Further, $B_1^\top \tilde{I} B_2 \neq 0$. So if $A_3 = [A_1, A_2]$, this means the skew-symmetric matrix C_3 is non-zero. Hence C_3 must have at least two columns, that is $\dim U_0 \geq 2$. Then

$$n = \dim U_0 + \dim W + \dim U_0^* \geq 2 + 4 + 2 = 8$$

holds. \square

Remark 5.2. With the additional assumption that the centralizer of Γ in $\text{Iso}(\mathbb{R}^{r,s})$ acts transitively, the second author has a proof (to appear in his dissertation) that the dimension bound in Theorem 5.1 can be improved to $n \geq 14$. As Example 6.4 shows, this is a sharp lower bound.

6. EXAMPLES

Lemma 6.1. *If the centralizer of Γ in $\text{Iso}(\mathbb{R}^{r,s})$ has an open orbit U , then the Γ -action preserves U , that is $\Gamma.U = U$.*

Proof. By taking the Zariski closure, we may assume from the beginning that Γ is an algebraic subgroup of $\text{Iso}(\mathbb{R}^{r,s})$. Since the elements of Γ are unipotent, the algebraic group Γ is also connected. The centralizer G of Γ is also an algebraic subgroup, and as such it has finitely many open orbits in $\mathbb{R}^{r,s}$ (cf. [1, Proposition 6.8]). The group Γ permutes the open orbits of G . Since it is connected, Γ , in fact, preserves each orbit. \square

Example 6.2. Let $\Gamma_{4,4} \subset \text{Iso}(\mathbb{R}^{4,4})$ be the group generated by

$$\gamma_1 = \left(\begin{pmatrix} I_2 & -B_1^\top \tilde{I} & 0 \\ 0 & I_4 & B_1 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} 0 \\ w_1 \\ 0 \end{pmatrix} \right), \quad \gamma_2 = \left(\begin{pmatrix} I_2 & -B_2^\top \tilde{I} & 0 \\ 0 & I_4 & B_2 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} 0 \\ w_2 \\ 0 \end{pmatrix} \right)$$

in the basis representation (4.3). Here,

$$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

and $\tilde{I} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ is the signature matrix of W . Their commutator is

$$\gamma_3 = [\gamma_1, \gamma_2] = \left(\begin{pmatrix} I_2 & 0 & C_3 \\ 0 & I_4 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ 0 \end{pmatrix} \right),$$

with

$$C_3 = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

One checks that $A_i^2 = 0$ and that γ_3 commutes with γ_1, γ_2 . Therefore, $\Gamma_{4,4}$ is isomorphic to the discrete Heisenberg group on two generators.

In the chosen basis, the pseudo-scalar product is represented by the matrix $Q = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & \tilde{I} & 0 \\ I_2 & 0 & 0 \end{pmatrix}$. The following elements $S \in \mathfrak{iso}(\mathbb{R}^{4,4})$, where $\mathfrak{iso}(\mathbb{R}^{4,4})$ denotes the Lie algebra of $\text{Iso}(\mathbb{R}^{r,s})$, commute with (A_1, v_1) and (A_2, v_2) :

$$S = \left(\begin{pmatrix} S_1 & -S_2^\top \tilde{I} & 0 \\ 0 & S_3 & S_2 \\ 0 & 0 & -S_1^\top \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right),$$

where $x = (x_1, x_2)^\top$, $y = (y_1, y_2, y_3, y_4)^\top$, $z = (z_1, z_2)^\top$ are arbitrary and

$$S_1 = \begin{pmatrix} z_1 & z_2 \\ z_2 & -z_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -y_1 & y_3 - y_2 \\ -y_2 & y_1 + y_4 \\ -y_3 & 0 \\ -y_4 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & -z_2 & -z_1 \\ 0 & 0 & z_1 & -z_2 \\ -z_2 & z_1 & 0 & 0 \\ -z_1 & -z_2 & 0 & 0 \end{pmatrix}.$$

Hence the elements $\exp(S)$ are contained in the centralizer of $\Gamma_{4,4}$ in $\text{Iso}(\mathbb{R}^{r,s})$. As x, y, z are arbitrary, the centralizer of $\Gamma_{4,4}$ has an open orbit U through the point 0. The set of all elements S is not a Lie subalgebra of the centralizer.

Corollary 6.3. *There exists a flat incomplete homogeneous pseudo-Riemannian manifold of signature $(4, 4)$ with non-abelian linear holonomy group.*

Proof. By Lemma 6.1 and Proposition 7.3, $\Gamma_{4,4}$ acts properly discontinuously and freely on every open orbit U of its centralizer in $\text{Iso}(\mathbb{R}^{4,4})$. So $M_{4,4} = U/\Gamma_{4,4}$ is a homogeneous manifold. The unit vector e_7 is a fixed point for $\gamma_3 \in \Gamma_{4,4}$, so the action of the centralizer is not transitive. Hence, $U \neq \mathbb{R}^{4,4}$, and M is incomplete. \square

Example 6.4. Let $\Gamma_{7,7} \subset \text{Iso}(\mathbb{R}^{7,7})$ be the group generated by

$$\gamma_1 = \left(\begin{pmatrix} I_5 & -B_1^\top \tilde{I} & C_1 \\ 0 & I_4 & B_1 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^* \end{pmatrix} \right), \quad \gamma_2 = \left(\begin{pmatrix} I_5 & -B_2^\top \tilde{I} & C_2 \\ 0 & I_4 & B_2 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2^* \end{pmatrix} \right)$$

in the basis representation (4.3). Here,

$$B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad u_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad u_2^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $\tilde{I} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ is the signature matrix of W . Their commutator is

$$\gamma_3 = [\gamma_1, \gamma_2] = \left(\begin{pmatrix} I_5 & 0 & C_3 \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ 0 \end{pmatrix} \right),$$

with

$$C_3 = \begin{pmatrix} 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

One checks that $A_i^2 = 0$ and that $\Gamma_{7,7}$ is isomorphic to a discrete Heisenberg group.

In the chosen basis, the pseudo-scalar product is represented by the matrix $Q = \begin{pmatrix} 0 & 0 & I_5 \\ 0 & \tilde{I} & 0 \\ I_5 & 0 & 0 \end{pmatrix}$. The following elements $S \in \mathfrak{iso}(\mathbb{R}^{7,7})$ commute with (A_1, v_1) and (A_2, v_2) :

$$S = \left(\begin{pmatrix} S_1 & -S_2^\top \tilde{I} & S_3 \\ 0 & 0 & S_2 \\ 0 & 0 & -S_1^\top \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right),$$

where $x = (x_1, \dots, x_5)^\top, y = (y_1, \dots, y_4)^\top, z = (z_1, \dots, z_5)^\top$ are arbitrary and

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -2z_2 \\ 0 & 0 & 0 & 0 & 2z_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & -z_2 & z_1 & 0 \\ 0 & 0 & z_1 & z_2 & 0 \\ 0 & 0 & -z_1 & z_2 & 0 \\ 0 & 0 & z_2 & z_1 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & 0 & -y_2 - y_3 & y_4 - y_1 & 0 \\ 0 & 0 & y_1 + y_4 & y_3 - y_2 & 0 \\ y_2 + y_3 & -y_1 - y_4 & 0 & z_5 & -z_4 \\ y_1 - y_4 & y_2 - y_3 & -z_5 & 0 & z_3 \\ 0 & 0 & z_4 & -z_3 & 0 \end{pmatrix}.$$

The linear part of such a matrix S is conjugate to a strictly upper triangular matrix via conjugation with the matrix

$$T = (e_1, e_2, e_3, e_4, e_7 + e_8, e_5, e_6, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_7 - e_8),$$

where e_i denotes the i th unit vector. Hence, the elements $\exp(S)$ generate a unipotent group of isometries whose translation parts contain all of \mathbb{R}^{14} . Therefore, the centralizer of $\Gamma_{7,7}$ in $\text{Iso}(\mathbb{R}^{7,7})$ acts transitively (see [1, Corollary 6.27] or [3, Theorem 4.2]). In particular, $\Gamma_{7,7}$ acts freely on $\mathbb{R}^{7,7}$.

It can be verified that the set of all matrices S forms a 3-step nilpotent Lie subalgebra of the centralizer algebra. Hence the set of all $\exp(S)$ forms a unipotent group of isometries acting simply transitively on $\mathbb{R}^{7,7}$.

Corollary 6.5. *There exists a flat complete homogeneous pseudo-Riemannian manifold of signature $(7, 7)$ with non-abelian linear holonomy group.*

Proof. By Proposition 7.2, the group $\Gamma_{7,7}$ acts properly discontinuously and freely on $\mathbb{R}^{7,7}$. So $M = \mathbb{R}^{7,7} / \Gamma_{7,7}$ is a complete homogeneous manifold. \square

7. PROPERNESS OF ACTIONS WITH TRANSITIVE CENTRALIZER

Recall that an action of a Lie group L on a locally compact Hausdorff space X is called *proper* if and only if for all compact sets $K \subset X$ the set $\{\ell \in L \mid \ell K \cap K \neq \emptyset\}$ is compact.

Lemma 7.1. *Let $X = G/H$ be a homogeneous space, where G is a Lie group and H is a closed subgroup. Let $L \leq \text{Diff}(X)$ be a group of diffeomorphisms of X which centralizes G . Then L acts properly on X if and only if L is a closed subgroup of $\text{Diff}(X)$ with respect to the compact open topology.*

Proof. Choose a basepoint $x_0 \in X$ such that $H = G_{x_0}$ is the stabilizer of x_0 . Then X is homeomorphic to G/H via the orbit map $o : G/H \rightarrow X, g \mapsto g \cdot x_0$. The right-action of $N_G(H)$ on G induces a continuous homomorphism onto the

centralizer $Z_X(G)$ of G in $\text{Diff}(X)$. Let \overline{L} denote the preimage of L in $N_G(H)$. In particular, if L is closed in $\text{Diff}(X)$ then \overline{L} is closed in G . Note that $X/L = G/\overline{L}$ is a Hausdorff space if and only if the subgroup \overline{L} is closed in G . Since L acts freely on X , X/L is Hausdorff if and only if L acts properly on X . This proves the lemma. \square

We can apply this criterion in the affine situation, as follows:

Proposition 7.2. *Let $L \leq \text{Aff}(\mathbb{R}^n)$ be a subgroup whose centralizer in $\text{Aff}(\mathbb{R}^n)$ acts transitively on \mathbb{R}^n . Then the action of L on \mathbb{R}^n is proper if and only if L is a closed subgroup of $\text{Aff}(\mathbb{R}^n)$.*

Similarly, assume that the centralizer G of L in $\text{Aff}(\mathbb{R}^n)$ has an open orbit $U = G \cdot x_0$ which is preserved by L . Then L acts freely on U , and the action is proper if and only if L is closed in $\text{Diff}(U)$. Since $\text{Diff}(U) \cap \text{Aff}(\mathbb{R}^n)$ is closed in $\text{Aff}(\mathbb{R}^n)$ (cf. [1, Lemma 6.9]), the above proposition generalizes to:

Proposition 7.3. *Let $L \leq \text{Aff}(\mathbb{R}^n)$ be a subgroup whose centralizer in $\text{Aff}(\mathbb{R}^n)$ acts transitively on an open subset U of \mathbb{R}^n . If L preserves U then the action of L on U is proper if and only if L is a closed subgroup of $\text{Aff}(\mathbb{R}^n)$.*

Remark 7.4. Püttmann [6, Section 4.2] gives an example of a free action of the abelian group $(\mathbb{C}^2, +)$ on \mathbb{C}^5 by unipotent affine transformations, such that the quotient is not a Hausdorff space. Hence the action is not proper.

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DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ALGEBRA AND GEOMETRY, KARLSRUHE INSTITUTE FOR TECHNOLOGY, 76131 KARLSRUHE, GEMANY

E-mail address: baues@kit.edu

DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ALGEBRA AND GEOMETRY, KARLSRUHE INSTITUTE FOR TECHNOLOGY, 76131 KARLSRUHE, GERMANY

E-mail address: globke@math.uni-karlsruhe.de